Hidden Circles and the Digits of Π

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The number π is defined as the quotient of a circular object's circumference with its diameter. In 5th or sixth grade (I can't quite remember) the teacher asked us to take round objects (coffee cups, pots, balls, whatever we could find) and do this measurement. In preparing this essay, I just repeated this age old exercise, with my own coffee cup and a little round device which measures the humidity in my office (a hygrometer). Table 1 shows what I measured with a primitive tape measure. C and d denote measured perimeter and diameter, q the computed quotient.

Object	C (cm)	d (cm)	q
Cup	27.7	8.8	3.14773
Hygrometer	26.4	8.4	3.14286

Table 1: Measurements

Now, as we all know that $\pi = 3.141592636....$ it is clear that my method didn't provide great accuracy. Of course, I could have tried harder, with a better tape measure and larger round objects. I could also have repeated my measurement many times and computed averages. I do, however, not wish to waste anybody's time here. The number π was shown to be between 223/71 and 22/7 already 2200 years ago, by Archimedes, who used regular polygons to approximate a circle (actually, a teacher in my high school taught us that Archimedes had already computed hundreds of digits, and I readily believed this until a referee for this article taught me otherwise; according to Wikipedia, π was known to less than 10 digits until the year 1,000 (1),(2)). There are now many algorithms one can use to approximate π, as well as experiments. We know that π is a transcendental number (meaning it is neither rational nor algebraic), a property which follows from what is known as the Lindemann-Weierstrass Theorem. In particular, the digits of π will never repeat periodically, nor can we expect any pattern.

Let me mention a second simple experiment which can be used to approximate π. In elementary mechanics one studies the pendulum, and after some

simplifications derives the relationship

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where T is the period and l is the length of the pendulum, and $g = 9.80665 \quad m/sec^2$ is standard gravity (earth acceleration of a free falling object at sea level). The appearance of π in this formula may appear a bit as a mystery... until you understand that the solution of the differential equation governing the pendulum motion approximates a circular motion in phase space, and circles, of course, are the underlying geometric objects. You could set the generic task: See π, find the circle.

It is a standard exercise in a physics lab to compute g by measuring T and l; conversely, if one already knows g (from, say, free fall experiments), one can use the formula to approximate π. However, no higher level accuracy need to be expected, because the formula itself is only an approximation (derived as a linear approximation from the true pendulum equations), reasonably accurate only for small amplitudes. In addition, measurement errors will affect the quantities T and l. If you wish to use this experiment to find π your pendulum should be a string of length l, with an attached weight that is much heavier than the string (because the formula is derived for such conditions).

I did actually do the experiment with a home-made pendulum (made of a string with a golf ball attached at one end). The length of this pendulum from pivot point to the center of the golf ball was 105 cm (I did the best I could measuring this), and this pendulum exhibited a frequency of 29 swings in 60 seconds. This gives $T = 2.068965...$ seconds, and inserting these data and the value of g into the formula, I found $\pi \approx 3.1614....$ I then lengthened the string, producing a pendulum with $l = 114.5$ cm, and measured that this pendulum swung 56 times in 120 seconds, or $T = 2.142857...$ seconds. This gives $\pi \approx 3.1346...$; the average of both experiments is 3.148. Of course, it's off, but this "accuracy" exceeded my expectations.

For practical purposes it should never be necessary to compute π to more than, say, 50 digits. However, there is some interest in the methods

themselves, as they tend to shed light on many truths (or hidden truths) in geometry and analysis.

Consider the well-known expansion, known as the Leibniz series,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots.$$

A bit of playing with your pocket calculator will convince you that this is not a very good method to compute the digits of π: the first four terms on the right, multiplied by 4, give 2.895238... One has to add many, many terms until reasonable accuracy is obtained (actually, you can read off the formula how many terms you have to include to compute π to, say, 100 digits. Give it a try). Nevertheless, the formula should intrigue your interest, because as written, there is no hint why this should be true at all. The hidden truth is that this formula arises from an elementary trigonometric identity, an integration and a so-called power series expansion, all things we do in first year calculus, but which were at the forefront of mathematical research some 250 years ago (Leonhard Euler, who would recently have celebrated his 306th birthday, did much of the research).

Here are the steps: We know that $\tan(\pi/4) = 1$ (a line at 45 degrees counterclockwise from the horizontal has slope 1, and this is where the circle is hiding), so $\pi/4 = \tan^{-1}1$. From Calculus we know that

$$\tan^{-1}1 = \int_0^1 \frac{1}{1+x^2} \, dx,$$

and if we use the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots.$$

and integrate term by term, with $\int_0^1 x^k \, dx = \frac{1}{k+1}$, we find exactly Leibniz' formula. Of course, we have to somehow justify the manipulations with infinitely many terms. This can be and is done in university textbooks on real analysis.

The above steps also show that

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \qquad (1)$$

which allows approximation of all values of the inverse tangent. Note that this works better (the series on the right converges faster) for smaller values of x. Further, the trigonometric identity

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

may be used (with some work) to derive the formula (attributed to a man named Machin, and first published in 1706)

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}. \qquad (2)$$

This offers a much better way to approximate π, because for $x = 1/5$ and $x = 1/239$ the right hand side of (1) converges much faster. Again, feel free to play with a pocket calculator: for example, the final displayed term in (1), evaluated for $x = 1/5$, is 0.0000018286... Much more on the power of Machin-like formulas to compute π with great accuracy can be found in (3; 4). In combination with modern supercomputers, these formulas allow accuracy to billions of digits.

A different and powerful formula is known as the Bailey-Borwein-Plouffe series

$$\pi = a_0 + \frac{1}{16}a_1 + \frac{1}{16^2}a_2 + \frac{1}{16^3}a_3 + \dots.$$

where

$$a_n = \frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}$$

for $n = 0, 1, 2, \dots.$ Adding just the first two terms on the right gives 3.141422466... This series converges to π at a very impressive rate. Two other things are rather remarkable about it: First, the derivation of this identity requires no more than clever high school algebra and elementary integrations (involving trigonometric functions; this is, again, where the circle is hiding) and could have been done centuries ago (but it was first published only in the 1990s (5)). Second, in the hexadecimal system (based on the base 16, vs. 10 for the decimal system), this formula allows computing digits of π without knowing all previous digits.

In passing, the number π also shows up in the identities

$$e^{i\pi} + 1 = 0$$

(which nicely combines five fundamental real and complex numbers), and

$$\int_0^\infty e^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi}$$

(which is of major importance in statistics). I leave it as a challenge to the reader to find "where the circle is hiding" in these identities.

So, I have shown you four ways of getting to π, two "experimental," two computational. With more effort, time and information, one could fill a

book with such methods (and, to be sure, most of its content is already on the internet). I will provide one more example, this time from a (hypothetical) billiard game. This was first pointed out by Gregory Galperin (6) and is a fairly recent observation.

Consider two fully elastic billiard balls, which move on a straight line (in one dimension), without gravitational forces or friction, and with fully elastic collisions (meaning that collisions between the balls preserve momentum and kinetic energy). We present a scenario in which there is a solid elastic wall at $x = 0$, ball A (with initial speed 0) has mass 1, radius 1 and is centered at $x = 3$, and ball B has mass $m \geq 1$, radius 1, velocity -1 and is centered at x=6 (the radii and initial centers are actually of no importance; what matters is that the balls are initially apart, and ball B is set up to hit ball A from the right at speed 1). See Figure 1.
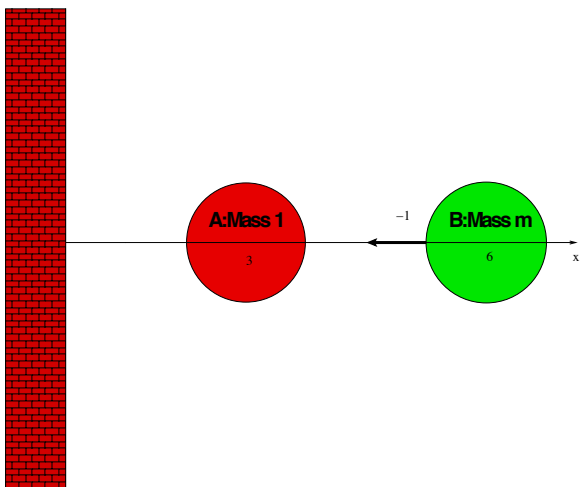

Fig. 1: Two elastic spheres

If $m = 1$, then the balls are of equal mass and it is very predictable what will happen: Ball B will hit ball A, they exchange velocities, then ball A hits the wall, bounces back with velocity $+1$, hits ball B a second time, and ball B flies off with velocity $+1$. We observe one wall collision and three collisions in all. What happens if ball B is heavier than ball A, meaning $m > 1$? Momentum and energy conservation still uniquely determine the outcome of each collision: If, say, we let u_0 be the initial velocity of ball A (we took $u_0 = 0$) and v_0 the initial velocity of ball B (we took $v_0 = -1$) then the velocities u_0', v_0' *after* the collision will be

$$u_0' = u_0 - \frac{2m}{m+1}(u_0 - v_0) \qquad (3)$$

$$v_0' = v_0 + \frac{2}{m+1}(u_0 - v_0) \qquad (4)$$

This is known as the collision transformation (a well-known concept in the kinetic theory of gases). One readily checks that

$$u_0' + mv_0' = u_0 + mv_0$$

(momentum conservation), and

$$(u_0')^2 + m(v_0')^2 = (u_0)^2 + m(v_0)^2 \qquad (5)$$

(energy conservation). The collision transformation is uniquely determined by these two properties.

Ball A will now bounce off the wall and head back right; it will collide again with ball B, but as ball B is heavier than ball A, this will not be the last collision—ball A will head for the wall again, bounce back, meet again with ball B, and so on. Figure 2 show what this looks like in x, t "space" time; for convenience, the particles have been shrunk to points (we mentioned before that the size of the particles did not matter).
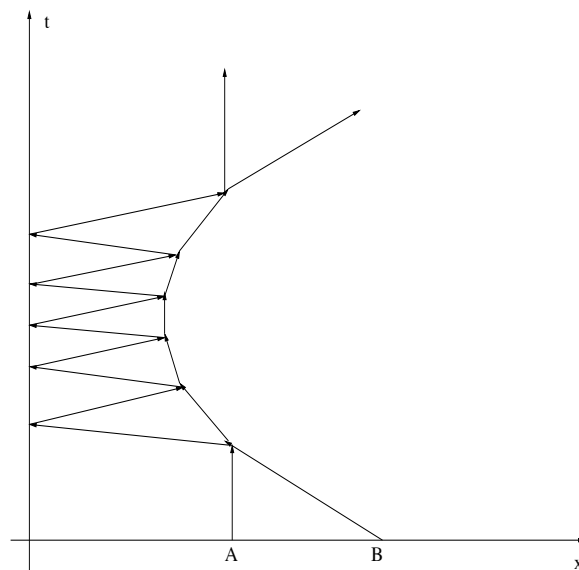

Fig. 2: Many collisions in spacetime

We need a little bit of terminology to carry on. Suppose that $u_0, u_1, u_2, \ldots$ denote the velocities of sphere A initially, after the first wall bounce, then after the second wall bounce, etc., and that $v_0, v_1, v_2, \ldots$ denote the velocities of sphere B initially, after the first collision with A, then after the second collision with A, etc. From the collision transformation we find $u_1 = -u_0', \quad v_1 = v_0'$, or

$$u_1 = \frac{m-1}{m+1}u_0 - \frac{2m}{m+1}v_0 \qquad (6)$$

$$v_1 = \frac{2}{m+1}u_0 + \frac{m-1}{m+1}v_0 \qquad (7)$$

The two particles were originally on collision course (or in a collision configuration) because $v_0 - u_0 = -1 < 0$, and if $v_1 - u_1 < 0$, they will collide again. We can then compute (u_2, v_2), (u_3, v_3) etc., until we find a number k such that, for the first time, $v_k - u_k > 0$. Particle A can then not catch up with particle B, and there will be no more collisions.

Using a computer, one can find the number k with a little bit of effort. Table 2 shows k as a function of m, the mass of particle B, and, following Galperin's idea, we have taken $m = 100^n$, where $n = 0, 1, 2, 3, \ldots$.

m	N (total)	M (wall touches)
1	3	1
100	31	15
10,000	314	157
10^6	3142	...
10^8	31415	...

Table 2: Number of collisions

Here, N and M are the numbers of total collisions and wall collisions, respectively. Remember that particle A is initially at rest, and particle B moves initially at $v_0 = -1$.

It appears that the number of all collisions (including the wall touches of ball A) produce the digits of π, and the number of wall touches is half or one less than half of the corresponding approximation of π. The latter is easily understood-it is possible that balls A and B have a final collision such that A retains a positive velocity and will not return to the wall (as sketched in 2). But why should this experiment produce the digits of π at all? Where is the circle hiding? Before we answer this riddle, let us point out that this "experiment" is really a thought experiment which you can't do in reality. Because, even to get just four digits of π you have to have ball B a million times heavier than ball A, and, of course, both have to be perfectly elastic and not be subject to friction or gravity.

But where is the circle? The explanation is hidden in the properties of the transformation (6,7), although another idea is necessary. This idea has to do with the identity (5), which provides the conservation of kinetic energy. It turns out that things become simpler if one rescales the speeds v_0, v_1, v_2 etc. of ball B by defining

$$w_0 = \sqrt{m}v_0, w_1 = \sqrt{m}v_1,$$

etc.

The energy conservation (5) becomes then the simpler equation

$$(u_0')^2 + (w_0')^2 = (u_0)^2 + (w_0)^2 \tag{8}$$

and the collision transformation (7) becomes

$$u_1 = \frac{m-1}{m+1}u_0 - \frac{2\sqrt{m}}{m+1}w_0 \tag{9}$$

$$w_1 = \frac{2\sqrt{m}}{m+1}u_0 + \frac{m-1}{m+1}w_0 \tag{10}$$

This is the same transformation as before, but the speed coordinate for ball B has been rescaled! In this new coordinate system, the equations (9,10) are where the circle is hiding: If you set

$$\alpha = \frac{m-1}{m+1}, \quad \beta = \frac{2\sqrt{m}}{m+1}$$

then one immediately checks that $\alpha^2 + \beta^2 = 1$, and therefore there is an angle θ such that $\cos\theta = \alpha, \sin\theta = \beta$. Geometrically this means that in the $u-w$ plane, (9,10) is a rotation in the counterclockwise sense by the angle θ, and in our setup we begin the rotation with the initial point $(0, -\sqrt{m})$. The speeds (u_j, w_j) computed from repeated application of (9, 10) arise from repeated rotations by θ in the $u-w$ plane for $j = 0, 1, 2, \ldots$, as shown in Figure 3, or as expressed by the transformation (rotation)

$$\begin{pmatrix} u_{j+1} \\ w_{j+1} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_j \\ w_j \end{pmatrix}$$

The energy conservation as stated in (8) is the key ingredient in this: It implies that the collision transformation in this context must conserve the length of the vector (u_0, w_0), and only rotations or reflections do this.
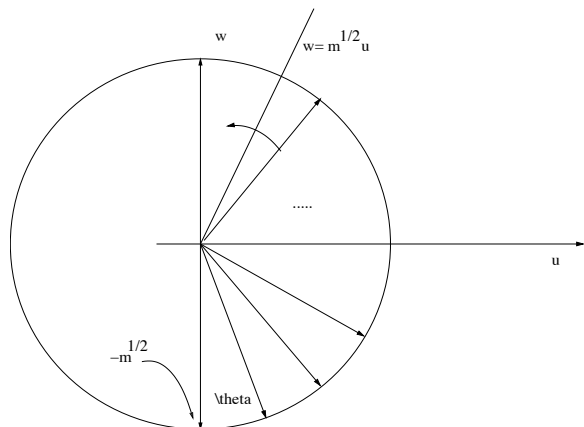

Fig. 3: Collisions are rotations!

We are almost there! There will be no more collisions after the first k for which $v_k > u_k$, or, equivalently, $w_k > \sqrt{m}u_k$. Hence we have to find out for which k the sum of the angles will have crossed the line with slope $\sqrt{m}$. From the picture, this means we are looking for the smallest k for which $\tan\left(k\theta - \frac{\pi}{2}\right) > \sqrt{m}$.

Now, let us consider a large m. Then $\tan^{-1}\sqrt{m} \approx \frac{\pi}{2}$ (or, there have been enough collisions to go almost through a half-circle, meaning $k\theta \approx \pi$.) We can also approximate θ in terms of m by observing that $\alpha = \cos\theta \approx 1 - \frac{\theta^2}{2}$, hence $\theta \approx \frac{2}{\sqrt{m+1}}$. Putting it all together gives $k \approx \pi\frac{\sqrt{m+1}}{2}$, and this is an approximation of the expected number of wall touches: For example, for $m = 10^4$, we find $2k \approx 100\pi \approx 314$.

There is much more about this on the internet, in particular in the article (6). I would like to thank my colleague Peter Dukes for bringing this way of finding π to my attention.

References

[1] http://en.wikipedia.org/wiki/Approximations of π

[2] http://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

[3] http://en.wikipedia.org/wiki/John Machin

[4] http://en.wikipedia.org/wiki/Machin-like formula

[5] http://en.wikipedia.org/wiki/BaileyBorwein Plouffe formula

[6] Gregory Galperin: Playing Pool with π (The Number π from a Billiard Point of View). Regular and Chaotic Dynamics 8(4), 2003. See $http://ics.org.ru/doc?pdf = 440$